# Computational Evolutionary Game Theory

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## Abstract

This chapter examines the intersection of evolutionary game theory and theoretical computer science. We will show how techniques from each field can be used to answer fundamental questions in the other. In addition, we will analyze a model that arises by combining ideas from both fields. First, we describe the classical model of evolutionary game theory and analyze the computational complexity of its central equilibrium concept. Doing so involves applying techniques from complexity theory to the problem of finding a game-theoretic equilibrium. Second, we show how agents using imitative dynamics, often considered in evolutionary game-theory, converge to an equilibrium in a routing game. This is an instance of an evolutionary game-theoretic concept providing an algorithm for finding an equilibrium. Third, we generalize the classical model of evolutionary game theory to a graph-theoretic setting. Finally, this chapter concludes with directions for future research. Taken as a whole, this chapter describes how the fields of theoretical computer science and evolutionary game theory can inform each other.

## 29.1 Evolutionary Game Theory

Classical evolutionary game theory models organisms in a population interacting and competing for resources. The classical model assumes that the population is infinite. It models interaction by choosing two organisms uniformly at random, who then play a 2-player, symmetric game. The payoffs that these organisms earn represent an increase or a loss in fitness, which either helps or hinders the organisms ability to reproduce. In this model, when an organism reproduces, it does so by making an exact replica of itself, thus a child will adopt the same strategy as its parent.

One of the fundamental goals of evolutionary game theory is to characterize which strategies are resilient to small mutant invasions. In the classical model of evolutionary game theory, a large fraction of the population, called the incumbents, all adopt the same strategy. The rest of the population, called the mutants, all adopt some other strategy. The incumbent strategy is considered to be stable if the incumbents retain a higher fitness than the mutants. Since the incumbents are more fit, they reproduce

more frequently and the fraction of mutants in the population will eventually go to 0. Put another way, an evolutionarily stable strategy (ESS) is a strategy such that if all the members of a population adopt it, then no mutant strategy could overrun the population. We shall see in Section 29.1.1 that ESS are a refinement of Nash equilibria.

Replication is not the only type of dynamic studied in evolutionary game theory. Imitation is another widely studied dynamic. In imitative dynamics, each agent initially plays some pure strategy. As time goes on, agents interact pairwise. After this pairwise interaction, if one agents sees the other agent earned a higher payoff, the agent with the lower payoff may adopt, or imitate, the strategy of the agent who earned the higher payoff. Imitative dynamics model, for example, a new idea, innovation, or fad spreading through a population of individuals or firms.

In general, there are two main characteristics common to most evolutionary game theoretic models. The first is that the population is infinite. The second is that players adopt a very simple, local dynamic, such as replication or imitation, for choosing and updating their strategies. These dynamics result in the agents learning from the other agents in their environment; they provide a method for an equilibrium strategy to emerge from the population. These types of dynamics explain *how* a population can converge to an equilibrium. For example, Section 18.3.1 shows that equilibria for the nonatomic selfish routing game exists, whereas Section 29.3 will show how agents obeying imitative dynamics can converge to it.

Next we will formally describe the basic model of evolutionary game theory. Then, in Section 29.2, we will analyze the computational complexity of finding and recognizing stable strategies. After that, in Section 29.3, we will see an example of imitative dynamics. We will apply imitative dynamics to the problem of selfish routing and show how agents converge to an equilibrium. Finally, in Section 29.4, we will examine the notion of stable strategies in a context where agents play against their local neighborhood in a graph, as opposed to playing against another agent chosen uniformly at random.

### 29.1.1  The Classical Model of Evolutionary Game Theory

The classical model of evolutionary game theory considers an infinite population of organisms, where each organism is assumed to be equally likely to interact with each other organism. Interaction is modeled as playing a fixed, 2-player, symmetric game defined by a fitness function $F$ (we emphasize that the same game $F$ is played in all interactions). Let $A$ denote the set of actions available to both players, and let $\Delta(A)$ denote the set of probability distributions or mixed strategies over $A$, then $F: \Delta(A) \times \Delta(A) \to \Re$. If two organisms interact, one playing a mixed strategy $s$ and the other playing a mixed strategy $t$, the $s$-player earns a fitness of $F(s|t)$ while the $t$-player earns a fitness of $F(t|s)$.

In this infinite population of organisms, suppose that there is a $1 - \epsilon$ fraction who play strategy $s$, and call these organisms *incumbents*, and suppose that there is an $\epsilon$ fraction who play $t$, and call these organisms *mutants*. Assume that two organisms are chosen uniformly at random to play each other. The strategy $s$ is an ESS if the expected fitness of an organism playing $s$ is higher than that of an organism playing $t$, for all $t \neq s$ and all sufficiently small $\epsilon$. Since an incumbent will meet another incumbent with probability $1 - \epsilon$ and it will meet a mutant with probability $\epsilon$, we can calculate the

expected fitness of an incumbent, which is simply $(1 - \epsilon)F(s|s) + \epsilon F(s|t)$. Similarly, the expected fitness of a mutant is $(1 - \epsilon)F(t|s) + \epsilon F(t|t)$. Thus we come to the formal definition of an ESS.

**Definition 29.1**      A strategy $s$ is an *evolutionarily stable strategy (ESS)* for the 2-player, symmetric game given by fitness function $F$, if for every strategy $t \neq s$, there exists an $\epsilon_t$ such that for all $0 < \epsilon < \epsilon_t$, $(1 - \epsilon)F(s|s) + \epsilon F(s|t) > (1 - \epsilon)F(t|s) + \epsilon F(t|t)$.

If one assumes that each organism reproduces asexually, and spawns a number of offspring proportional to its fitness, then stable strategies will be those where the incumbent population will reproduce more than any small mutant invasion. Thus the mutant invasion will have fewer offspring and, in the long run, the fraction of mutants in the population will tend to 0. In fact, a continuous time analysis of the replicator dynamics shows that every ESS is asymptotically stable.

Definition 29.1 holds if and only if either of two conditions on $s$ is satisfied $\forall t \neq s$: (1) $F(s|s) > F(t|s)$, or (2) $F(s|s) = F(t|s)$ and $F(s|t) > F(t|t)$. A consequence of this alternate formulation of an ESS is that for $s$ to be an ESS, it must be the case that $F(s|s) \geq F(t|s)$, for all strategies $t$. This inequality means that $s$ must be a best response to itself, and thus for any ESS $s$, the strategy profile $(s, s)$ must also be a Nash equilibrium. This results in another, equivalent way to define an ESS.

**Theorem 29.2**     *A strategy s is an ESS for a 2-player, symmetric game given by fitness function F, if and only if $(s, s)$ is a Nash equilibrium of F, and for every best response t to s, where $t \neq s$, $F(s|t) > F(t|t)$.*

In general the notion of ESS is more restrictive than Nash equilibrium, and not all 2-player, symmetric games have an ESS.

Next, we give an example of a 2-player, symmetric game called Hawks and Doves, and then solve for its ESS. The game of Hawks and Doves models two organisms fighting over a resource. Obtaining the resource results in a gain of fitness of $V$, while fighting for the resource and losing results in a fitness decrease of $C$. If a Hawk plays a Dove, the Hawk will fight for the resource and the Dove will give up. This results in a Hawk earning in increase of fitness of $V$, and the Dove's fitness staying the same. If two Doves play each other, they split the resource earning them both a fitness increase of $V/2$. If two Hawks play, eventually one will win and one will lose, and it assumed that each organism has a 1/2 chance of being the winner. Figure 29.1 shows the payoff matrix for this game.

The strategy profile $(D, D)$ is not a Nash Equilibrium because one player could unilaterally deviate and play $H$ and increase its payoff from $V/2$ to $V$. Since $(D, D)$ is

$$
\begin{array}{c|cc}
 & H & D \\
\hline
H & (V - C)/2 & V \\
D & 0 & V/2
\end{array}
$$

**Figure 29.1.** The game of Hawks and Doves.

not a Nash Equilibrium, $D$ cannot be an ESS. Now, if $V > C$ then $H$ is an ESS. To see this observe that $F(H|H) = (V - C)/2$. Let $t$ be any mixed strategy with probability $p < 1$ of playing $H$ and $1 - p$ of playing $D$, then $F(t|H) = p\frac{V-C}{2} + (1 - p)0 < (V - C)/2$. Since $F(H|H) > F(t|H)$ for all $t \neq H$, $H$ is an ESS. We leave it as an exercise for the reader (see Section 29.6) to show that if $V \leq C$, the mixed strategy of playing $H$ with probability $V/C$ and $D$ with probability $1 - V/C$ is an ESS. Observe that as $V \to C$, the probability of playing $H$ approaches 1. This coincides with the pure strategy ESS of playing $H$ when $V > C$.

## 29.2 The Computational Complexity of Evolutionarily Stable Strategies

Next we show the computational complexity of finding an ESS given a 2-player symmetric game is both NP-hard and coNP-hard. To prove this, we will make a reduction from the problem of checking if a graph has a maximum clique of size exactly $k$. Prior work has shown that this problem is both NP-hard and coNP-hard. Along the way to proving the hardness of finding an ESS, we will see that the problem of recognizing whether a given strategy is an ESS is also coNP-hard.

Next we will give the intuition behind the reduction. The reduction will transform a graph $G$ into a payoff matrix $F$ which will have an ESS if and only if the size of the largest clique in $G$ is not equal to $k$. The reduction transforms the adjacency matrix of $G$ into the payoff matrix $F$ by replacing all the diagonal entries with the value 1/2, inserting a 0th row with each entry having a constant value, and inserting a 0th column with each entry having the same constant value.

Informally speaking, for a mixed strategy $s$ to be an ESS, incumbents should receive a relatively high payoff when playing other incumbents. In order for a strategy $s$ to have this property for the game $F$, when $s$ plays itself it must guarantee that the pure strategies chosen will correspond to two adjacent vertices. One can see that having a mixed strategy with support over a clique will achieve this. We will show in Lemma 29.3 that having support over a clique will result in a higher payoff than having support over a dense subgraph that is not a clique. Having the diagonal entries consist of the constant 1/2 will help us prove this. This lemma will allow us to prove that when the size of the maximum clique is greater than $k$, the uniform mixed strategy corresponding to vertices of the clique will be an ESS. In addition, setting the 0th row and column of $F$ to a carefully chosen constant will give us a pure strategy ESS in the case where the size of the maximum clique is less than $k$. This constant will also allow us to show that there is no ESS in the case where the size of the maximum clique in $G$ is exactly $k$.

In describing this reduction, and for the rest of this chapter, we use the notation $F(s|t)$ to denote the payoff of the player playing strategy $s$ when confronted with a player playing strategy $t$. When we are referring to a specific entry in the payoff matrix of $F$, we will use the notation $F(i, j)$ to denote the entry in the $i$th row and $j$th column. Also, if $s$ is a mixed strategy, we let $s_i$ denote the probability that the pure strategy $i$ is played. (Thus we will use $s$ and $t$ to denote mixed strategies, and $i$ and $j$ to denote indices into these mixed strategies, as well as indices into the payoff matrix $F$.)

The reduction from a graph $G = (V, E)$ to a payoff matrix $F$ that we consider works as follows.

- for $1 \leq i \neq j \leq n$: $F(i, j) = 1$ if $(i, j) \in E$ and $F(i, j) = 0$ if $(i, j) \notin E$
- for $1 \leq i \leq n$: $F(i, i) = 1/2$
- for $0 \leq i \leq n$: $F(0, i) = F(i, 0) = 1 - 1/(2k)$

To show that $F$ has an ESS if and only if the size of the largest clique in $G$ its not equal to $k$, we will need the following technical lemma.

**Lemma 29.3**    *If $s$ is a strategy with $s_0 = 0$, then $F(s|s) \leq 1 - 1/(2k')$, where $k'$ is the size of the maximum clique in $G$. This holds with equality if and only if $s$ is the uniform distribution over a $k'$-clique.*

**PROOF**    The proof is by induction on the number of nonedges between the vertices in $G = (V, E)$ corresponding to elements of the support set of $s$. The base case is when there are 0 such non-edges, which means the vertices corresponding to the support set of $s$ form a $k''$-clique, where $k'' \leq k$. We assume, without loss of generality, that the vertices in the $k''$-clique are numbered $1, 2, \ldots, k''$.

$$F(s|s) = \sum_{i \in [k'']} \sum_{j \in [k'']} s_i s_j F(i, j)$$

$$= \sum_{i \in [k'']} \sum_{j \in [k'']} s_i s_j - \sum_{i \in [k'']} s_i^2/2$$

$$= \sum_{i \in [k'']} s_i \sum_{j \in [k'']} s_j - 1/2 \sum_{i \in [k'']} s_i^2$$

$$\leq 1 - 1/(2k'')$$

The last inequality comes from the fact that when $||s||_1 = 1$, $||s||_2$ is minimized, and the inequality is tight, only when all of the components of $s$ are equal. Conversely, if $s$ is the uniform distribution over a $k'$-clique then, the inequality is tight, which is shown as follows,

$$\sum_{i \in [k']} \sum_{j \in [k']} s_i s_j F(i, j) = 1/k'^2 \sum_{i \in [k']} \sum_{j \in [k']} F(i, j)$$

$$= 1/k'^2 [k'^2 - k'/2]$$

$$= 1 - 1/(2k').$$

For the inductive step, let $u$ and $v$ be two vertices such that $(u, v) \notin E$. We construct a new strategy $s'$ by moving the probability from $v$ to $u$. So let $s'_u = s_u + s_v$ and $s'_v = 0$, and let the rest of the values of $s'$ be identical to those of $s$. Since $v$ is no longer in the support set of $s$, we can use the induction hypothesis to conclude that $F(s'|s') \leq 1 - 1/(2k')$. Let $p = \sum_{(u,w) \in E} s_w$ and let $q = \sum_{(v,w) \in E} s_w$, and without loss of generality assume that $p \geq q$. By writing out the expressions for $F(s'|s')$ and $F(s|s)$ one can show $F(s'|s') = F(s|s) + 2s_v(p - q) + s_u s_v > F(s|s)$. Thus, $F(s|s) \leq 1 - 1/(2k')$, which proves the inductive step.    □

Now we will use this lemma to prove the necessary properties of the reduction. The next two lemmas, when taken together, show that if the maximum size clique in $G$ has size not equal to $k$, then $F$ has an ESS.

**Lemma 29.4**    *If $C$ is a maximal clique in $G$ of size $k' > k$, and $s$ is the uniform distribution on $C$, then $s$ is an ESS.*

**PROOF**    By Lemma 29.3, $F(s|s) = 1 - 1/(2k')$. By the construction of the pay-off matrix $F$, $F(0|s) = 1 - 1/(2k) < F(s|s)$. Also, for any $u \notin C$, $u$ is connected to at most $k' - 1$ vertices in $C$, thus $F(u|s) \leq 1 - 1/k' < F(s|s)$. Thus any best response to $s$ must have support only over $C$. Furthermore, by Lemma 29.3 the payoff of $s$ against $s$ is maximized when $s$ is the uniform distribution over $C$. Thus, $s$ is a best response to itself. To prove that $s$ is an ESS, it remains to show that for all $t \neq s$, that are best responses, to $s$, $F(s|t) > F(t|t)$. Again by Lemma 29.3, $F(t|t) < 1 - 1/(2k')$. Since $C$ is a clique and $s$ and $t$ are distributions with support over $C$, using the structure of $F$ one can compute that $F(s|t) = 1 - 1/(2k')$. Thus, $F(s|t) > F(t|t)$ and $s$ is an ESS.   □

**Lemma 29.5**    *If the maximum size clique in $G$ is of size $k' < k$ then the pure strategy 0 is an ESS.*

**PROOF**    For any mutant strategy $t$, $F(t|0) = 1 - 1/(2k) = F(0|0)$, thus 0 is a best response to itself. Next, we show that for any $t$ not equal to the pure strategy 0, $F(0|t) > F(t|t)$. To do so, we first show that we can assume that $t$ places no weight on the pure strategy 0. Let $t^*$ be the strategy $t$ with the probability of playing the pure strategy 0 set to the value 0 and then renormalized. So, $t_0^* = 0$ and for $i \neq 0$, $t^* = t_i/(1 - t_0)$. By writing out the expressions for $F(t|t)$ and $F(t^*|t^*)$, one can show $F(t|t) = (2t_0 - t_0^2)(1 - 1/(2k)) + (1 - 2t_0 + t_0^2)F(t^*|t^*)$. Since $F(0|t) = 1 - 1/(2k)$, $F(0|t) > F(t|t)$ if and only if $F(0|t) > F(t^*|t^*)$. Next, since the maximum size clique in $G$ has size $k' < k$, applying Lemma 29.3 gives $F(t^*|t^*) \leq 1 - 1/(2k') < 1 - 1/(2k) = F(0|t)$.   □

The next two lemmas, when combined, show that if the maximum size clique in $G$ has size exactly $k$, then $F$ has no ESS.

**Lemma 29.6**    *If the maximum size clique of $G$ is at least $k$, then the pure strategy 0 is not an ESS.*

**PROOF**    Since $F(0|0) = F(t|0) = 1 - 1/(2k)$ for any strategy $t$, the pure strategy 0 is a best response to itself. But, if $t$ is the uniform distribution on the maximum clique of $G$, which has size $k' \geq k$, then by Lemma 29.3 $F(t|t) = 1 - 1/(2k') \geq F(0|t)$. By Theorem 29.2, this means the pure strategy 0 cannot be an ESS.   □

**Lemma 29.7**    *If the maximum size clique of $G$ is at most $k$, then any strategy for $F$ that is not equal to the pure strategy 0, is not an ESS for $F$.*

The proof of this lemma uses techniques similar to those used in Lemmas 29.5 and 29.6, so we leave it as an exercise for the reader (see Section 29.6).

Taking Lemmas 29.4, 29.5, 29.6, and 29.7 together, we get the following theorem.

**Theorem 29.8**    *Given a 2-player, symmetric game F computing whether or not F has an ESS is both NP-hard and coNP-hard.*

Combining Lemmas 29.5 and 29.6 shows that it is coNP-hard to check whether a given strategy is an ESS or not.

**Theorem 29.9**    *Given a 2-player, symmetric game F and a strategy s, it is coNP-hard to compute whether or not s in an ESS of F.*

**PROOF**    Lemmas 29.5 and 29.6 imply that $G$ has maximum clique of size less than $k$ if and only if the pure strategy 0 is an ESS of $F$. Since the problem of determining whether a graph has a maximum clique of size less than $k$ is coNP-hard, the problem of recognizing an ESS is also coNP-hard.    □

Theorems 29.8 and 29.9 imply that *there exist* games for which, in all likelihood, efficient algorithms for finding and recognizing ESS do not exist. These results are important because if finding an ESS for a given class of games is NP-hard, it is unlikely that a finite population obeying some simple dynamic will quickly converge to it. But, this observation does not mean that one should avoid using models based on ESS. It simply means that to ensure the plausibility of a finite population model, one should check whether it is computationally tractable to find the ESS of the games the model considers. Moreover, this result does not directly imply that an *infinite* population, however, cannot quickly converge to an equilibrium. In fact, the next section explores the convergence time of an infinite population to an equilibrium.

## 29.3  Evolutionary Dynamics Applied to Selfish Routing

In this section we describe a method for applying evolutionary dynamics to the problem of selfish routing. The model will consider an infinite population of agents, each of which carries an infinitesimally small amount of flow in a network. The agents actions allow them to change the path that they traverse; however, agents will not be allowed to change their paths arbitrarily. The space of actions available to these agents will be governed by simple, imitative dynamics. We show how agents selfishly seeking out low latency paths, while obeying these imitative dynamics, converge to an approximate equilibrium. First, we will formally describe the model which is similar to the nonatomic selfish routing model shown in Section 18.2.1. Then, we will briefly outline a technique that shows, in the limit, these dynamics converge to an equilibrium. Finally, we will analyze the time of convergence to an approximate equilibrium.

### 29.3.1  The Selfish Routing Model with Imitative Dynamics

Let $G = (V, E)$ be a network with latency functions $l_e: [0, 1] \to \Re$ defined over each edge. We assume the latency functions are nonnegative, nondecreasing, and Lipschitz

continuous. We also assume that there is one unit of flow that is to be routed from a source $s$ to a sink $t$, and we let $P$ denote the set of $s$-$t$ paths in $G$. We also assume that there are infinitely many agents, each of which carries an infinitesimally small amount of flow. Let $x_p$ denote the fraction of flow that is being routed over path $p$. Thus the vector $\vec{x}$, which is indexed by the paths in $P$, will describe the flow over $G$ at a given point in time. A flow $\vec{x}$ is *feasible* if it routes 1 unit of flow from $s$ to $t$. Let $x_e = \sum_{p \ni e} x_p$ be the total *load* of an edge. The total *latency* of an edge is denoted $l_e(x_e)$ and the total latency of a path is the sum of the latencies of the edges in the path, $l_p(\vec{x}) = \sum_{e \in p} l_e(x_e)$. Finally, the average latency of the entire network is $\bar{l} = \sum_{p \in P} x_p l_p(\vec{x})$.

Initially each agent is assumed to play an arbitrary pure strategy. Then at each point in time, each agent is randomly paired with another agent and they compare the latencies of their paths. If the latency of one agent's path is less than the latency of the other agent's path, the agent experiencing higher latency switches to the lower latency path with probability proportional to the difference in latencies. These imitative dynamics model a source node gathering statistics on how long it takes for its packets to reach the destination and changing the route accordingly. In Section 29.3.2 we will describe why these dynamics will continue until the agents reach a Nash flow (also called Wardrop equilibrium), which is a pure strategy Nash equilibrium for this routing game, that we define next.

**Definition 29.10** A feasible flow $\vec{x}$ is a Nash flow if and only if for all $p$, $p' \in P$ with $x_p > 0$, $l_p(\vec{x}) \leq l_{p'}(\vec{x})$.

This definition ensures that, at a Nash flow, all $s$–$t$ paths have the same latency (this is precisely Definition 18.1 when restricted to the single commodity case). If we further restrict the latency functions to be strictly increasing, then Nash flows are essentially ESS. We omit the proof of this since this section focuses on the convergence of the imitative dynamics (we refer the interested reader to Section 29.6 for the appropriate references).

To analyze the convergence of these dynamics to either a Nash flow or an approximate equilibrium, it is necessary to compute the rate of change of the amount of flow over each path. Throughout this section we will use the notation $x'$ to denote the derivative with respect to time of the variable $x$, that is, $x' = dx/dt$. The following set of differential equations describe the rate of change of the flow over each path.

$$
\begin{aligned}
x'_p &= -x_p \sum_{q \in P: l_q(\vec{x}) < l_p(\vec{x})} x_q \lambda(\vec{x})[l_p(\vec{x}) - l_q(\vec{x})] \\
&\quad + \sum_{q \in P: l_q(\vec{x}) > l_p(\vec{x})} x_p x_q \lambda(\vec{x})[l_q(\vec{x}) - l_p(\vec{x})] \qquad (29.1) \\
&= \sum_{q \in P} x_p x_q \lambda(\vec{x})[l_q(\vec{x}) - l_p(\vec{x})] \\
&= \lambda(\vec{x}) x_p \left[ \sum_{q \in P} x_q l_q(\vec{x}) - l_p(\vec{x}) \sum_{q \in P} x_q \right] \\
&= \lambda(\vec{x}) x_p [\bar{l}(\vec{x}) - l_p(\vec{x})] \qquad (29.2)
\end{aligned}
$$

In this derivation, the function $\lambda$ accounts for normalizing factors so that the probabilities are bounded above by 1, and it accounts for the rate at which organisms are paired. The first summation in Equation 29.1 represents the expected number of agents that switch from path $p$ to lower latency paths. The probability than an agent on path $p$ is paired with an agent of path $q$ is equal to the fraction of agents using $q$, which is $x_q$. Then the agent using $p$ would switch to $q$ with probability $l_p(\vec{x}) - l_q(\vec{x})$. Multiplying this product by $x_p$ gives the expected number of agents moving from $p$ to a lower latency path $q$. Similarly, the second summation of Equation 29.1 represents the number of agents that switch to path $p$ from a higher latency path. The rest of the derivation results from straightforward algebraic manipulations.

Intuitively, Equation 29.2 says that paths with below average latency will have more agents switching to them than from them; paths with above average latency will have more agents switching from them than to them. In Section 29.3.3, where we bound the time it takes for the system to converge to an approximate equilibrium, we would like the rate of change of the population to be independent of the scale of the latency functions. Thus we will replace $\lambda(\vec{x})$ by $\bar{l}(\vec{x})^{-1}$ to give a relative rate of change.

While these equations resulted from imitative dynamics, the same equations can be derived from a type of replication dynamic. In the literature, these equations are often called the replicator dynamics. Now that we have defined the model and the dynamics, we will show that the population of agents using imitative dynamics will converge to an approximate equilibrium.

### 29.3.2  Convergence to Nash Flow

It has been shown that as time goes to infinity, any initial flow that has support over all paths in $P$ will eventually converge to a Nash flow. In this section we give an overview of the technique used to prove this. It is not clear how these techniques could yield a bound on the time to convergence, so we do not go into specific details of the proof. Since this text is focused on *algorithmic* game theory, we shall instead give more attention to another result, shown in Section 29.3.3, that bounds the time of convergence to an approximate equilibrium.

The main vehicle for proving that imitative dynamics converge to a Nash flow is Lyapunov's direct method. This is a general framework for proving that a system of differential equations converges to a stable point, without necessarily knowing how to solve the system of differential equations. Intuitively, this method works by first defining a real valued potential function $\Phi$ that measures the potential energy of the system of differential equations. The direct method requires that $\Phi$ be defined around a neighborhood of a stable point and vanish at the stable point itself. Then, if one can show that the dynamics of the system cause the potential function to decrease with respect to time (along with a few other technical properties of the potential function), Lyapunov's theorems will imply that if the system reaches the neighborhood of the stable point, the system will converge to the stable point. One drawback to this method is that it provides no guidance for choosing such a potential function.

The argument that applies this method to the system of differential equations described in Equation 29.2 works as follows. First, define $\Phi$ over the current flow such that it will measure the total amount of latency the agents are experiencing. We will

define just such a function in the next section. Then, show that the imitative dynamics cause $\Phi$ to decrease over time, and that $\Phi$ will achieve its minimum value at a Nash flow. Applying one of the theorems in the Lyapunov's framework allows one to conclude that if the dynamics ever reach a neighborhood of an equilibrium, they will converge to it. Finally, one has to show this neighborhood of convergence contains any initial, feasible flow with support over all paths in $P$. This comes from the fact that the dynamics cause the potential of any nonequilibrium flow to decrease and thus move toward an equilibrium. Thus, in this model of selfish routing with imitative dynamics, the Lyapunov framework allows one to show that the system will not get stuck in any local minima and will converge to global minimum from any initial state with support over all paths in $P$.

### 29.3.3 Convergence to Approximate Equilibrium

In this section we will give a bound on how long it takes for the population of agents using imitative dynamics to come to an approximate equilibrium.

One might consider using Euclidean distance between the current flow and an equilibrium flow as a measure of approximation. To see intuitively why this is not a suitable metric, consider a network and a flow where an $\epsilon$ fraction of the agents uses a path $p$, which has a latency that is slightly less than the current average latency. If it were essential for an equilibrium to have a large fraction of the population using $p$, we could take $\epsilon$ to be arbitrarily small, which, by Equation 29.2, means we could make $x_p'$ arbitrarily small. Thus the imitative dynamics would cause the population to move arbitrarily slowly to $p$, and therefore it would take arbitrarily long for the population to approach, in Euclidean distance, a Nash flow. Thus, we define an $\epsilon$-approximate equilibrium next.

> **Definition 29.11**      Let $P_\epsilon$ be the paths that have latency at least $(1 + \epsilon)\bar{l}$, that is $P_\epsilon = \{p \in P \mid l_p(\vec{x}) \geq (1 + \epsilon)\bar{l}\}$, and let $x_\epsilon = \sum_{p \in P_\epsilon} x_p$ be the fraction of agents using these paths. A population $\vec{x}$ is said to be at an $\epsilon$-approximate equilibrium if and only if $x_\epsilon \leq \epsilon$.

This definition ensures at such an equilibrium that only a small fraction of agents experience latency significantly worse than the average latency. In contrast, the definition of a Nash flow requires that all agents experience the same latency (see Definition 29.10).

To prove the convergence of these imitative dynamics to an approximate equilibrium, we will make use of the following potential function. This function is one way to measure the total amount of latency the agents experience.

$$\Phi(\vec{x}) = l^* + \sum_{e \in E} \int_0^{x_e} l_e(u)du \tag{29.3}$$

The integral sums the latency each agent that traverses edge $e$ would experience if the agents were inserted one at a time. Summing this over each edge gives the total latency that each agent would experience if they were entered into the network one at a time. The term $l^*$ denotes the minimum average latency of a feasible flow, $l^* = \min_{\vec{x}} \bar{l}$. We add this term as a technicality that will help prove our bounds on the time convergence

to approximate equilibrium. With the exception of the $l^*$ term, this is the same potential function described in Equation 18.3.

**Theorem 29.12**  *The imitative dynamics converge to an $\epsilon$-approximate equilibrium within time $O(\epsilon^{-3} \ln(l_{\max}/l^*))$.*

This proof works by analyzing the rate of change of $\Phi$ under the imitative dynamics. If the current flow is not at an $\epsilon$-approximate equilibrium, we can lower bound the absolute rate of change of $\Phi$ in terms of $\bar{l}$. We then lower bound $\bar{l}$ in terms of $\Phi$, resulting in a differential inequality. Solving it leads to an upper bound on the time it takes for $\Phi$ reach an approximate equilibrium.

**PROOF**  We start by computing the derivative with respect to time of the potential function $\Phi$.

$$\Phi' = \sum_{e \in E} x'_e l_e(x_e) = \sum_{e \in E} \sum_{p \ni e} x'_p l_e(x_e)$$

Next we substitute in the imitative dynamics, given by Equation 29.2. After that we simplify the expression with the aim of using Jensen's inequality.

$$\Phi' = \sum_{e \in E} \sum_{p \ni e} \lambda(\vec{x}) x_p [\bar{l}(\vec{x}) - l_p(\vec{x})] l_e(x_e)$$

$$= \lambda(\vec{x}) \sum_{p \in P} \sum_{e \in p} x_p [\bar{l}(\vec{x}) - l_p(\vec{x})] l_e(x_e)$$

$$= \lambda(\vec{x}) \sum_{p \in P} x_p [\bar{l}(\vec{x}) - l_p(\vec{x})] l_p(x_p)$$

$$= \lambda(\vec{x}) \left( \bar{l}(\vec{x}) \sum_{p \in P} x_p l_p(x_p) - \sum_{p \in P} x_p l_p(\vec{x})^2 \right)$$

$$= \lambda(\vec{x}) \left( \bar{l}(\vec{x})^2 - \sum_{p \in P} x_p l_p(\vec{x})^2 \right) \tag{29.4}$$

Jensen's inequality shows that this equation is bounded above by 0.

We would like to upper bound $\Phi'$. To do so, first observe as long as $\vec{x}$ is not at an $\epsilon$-approximate equilibrium, by definition at least an $\epsilon$ fraction of the population experiences latency at least $(1 + \epsilon)\bar{l}(\vec{x})$. Jensen's inequality also shows that for a fixed value of $\bar{l}(\vec{x})$, the $\sum_{p \in P} x_p l_p(\vec{x})^2$ term is minimized when the less expensive paths all have equal latency which we denote $l'$. Thus, for the purposes of upper bounding $\Phi'$, we assume $\bar{l} = \epsilon(1 + \epsilon)\bar{l} + (1 - \epsilon)l'$. Plugging this into Equation 29.4 gives

$$\Phi' \leq \lambda(\vec{x})[\bar{l}(\vec{x})^2 - (\epsilon((1 + \epsilon)\bar{l}(\vec{x}))^2 + (1 - \epsilon)l'^2)].$$

Now we substitute in $l' = \bar{l}\frac{1-\epsilon-\epsilon^2}{1-\epsilon}$ and perform some arithmetic giving,

$$\Phi' \leq -\lambda(\vec{x})\frac{\epsilon^3}{1-\epsilon}\bar{l}(\vec{x})^2$$

$$\leq -\lambda(\vec{x})\frac{\epsilon^3}{2}\bar{l}(\vec{x})^2.$$

We also replace $\lambda(\vec{x})$ with $\bar{l}(\vec{x})^{-1}$ to measure the relative rate of change of $\Phi$ under the imitative dynamics,

$$\Phi' \leq -\frac{\epsilon^3}{2}\bar{l}(\vec{x}). \tag{29.5}$$

We can bound $\bar{l}$ from below by $\Phi/2$ in the following way,

$$\bar{l}(\vec{x}) = \sum_{p \in P} x_p l_p(\vec{x}) = \sum_{p \in P} x_p \sum_{e \in p} l_e(x_e)$$

$$= \sum_{e \in E} \sum_{p \ni e} x_p l_e(x_e) = \sum_{e \in E} x_e l_e(\vec{x})$$

$$\geq \sum_{e \in E} \int_0^{x_e} l_e(u) du. \tag{29.6}$$

The inequality holds because of the assumed monotonicity of the latency functions. Now by the definition of $l^*$, it is easy to see that $\bar{l} \geq l^*$. Combining this fact with Equation 29.6, we get that $\bar{l} + \bar{l} \geq l^* + \sum_{e \in E} \int_0^{x_e} l_e(u) du = \Phi$. Thus $\bar{l} \geq \Phi/2$. Substituting this into Inequality 29.5, we get the following differential inequality,

$$\Phi' \leq -\epsilon^3 \Phi/4.$$

It can be shown via standard methods that any function of the following form is a solution to the above inequality,

$$\Phi(t) \leq \Phi(0)e^{-\epsilon^3 t/4}.$$

Here $\Phi(0)$ is given by the initial boundary conditions. Recall that this inequality only holds as long as $\vec{x}$ is not an $\epsilon$-approximate equilibrium. Thus, $\vec{x}$ must reach an $\epsilon$-approximate equilibrium when $\Phi$ reaches its minimum, $\Phi^*$, at the latest. So we find the smallest $t$ such that $\Phi(t) \leq \Phi^*$,

$$t = 4\epsilon^{-3} \ln \frac{\Phi(0)}{\Phi^*}.$$

It is easy to see that $\Phi^* \geq l^*$ and $\Phi(0) \leq 2l_{\max}$, which proves the theorem. □

## 29.4 Evolutionary Game Theory over Graphs

Next, we will consider a model similar to the classical model of evolutionary game theory described in Section 29.1, but we will no longer assume that two organisms are chosen uniformly at random to interact. Instead, we assume that organisms interact only with those in their local neighborhood, as defined by an undirected graph or network.

As in the classical setting (which can be viewed as the special case of the complete network or clique), we shall assume an infinite population, by which we mean we examine limiting behavior in a family of graphs of increasing size.

Before giving formal definitions, some comments are in order on what to expect in moving from the classical to the graph-theoretic setting. In the classical (complete graph) setting, there exist many symmetries that may be broken in moving to the network setting, at both the group and individual level. Indeed, such asymmetries are the primary interest in examining a graph-theoretic generalization.

For example, at the group level, in the standard ESS definition, one need not discuss any particular set of mutants of population fraction $\epsilon$. Since all organisms are equally likely to interact, the survival or fate of any specific mutant set is identical to that of any other. In the network setting, this may not be true: some mutant sets may be better able to survive than others due to the specific topologies of their interactions in the network. For instance, foreshadowing some of our analysis, if $s$ is an ESS but $F(t|t)$ is much larger than $F(s|s)$ and $F(s|t)$, a mutant set with a great deal of "internal" interaction (i.e., edges between mutants) may be able to survive, whereas one without this may suffer. At the level of individuals, in the classical setting, the assertion that one mutant dies implies that all mutants die, again by symmetry. In the network setting, individual fates may differ within a group all playing a common strategy. These observations imply that in examining ESS on networks we face definitional choices that were obscured in the classical model.

If $G$ is a graph representing the allowed pairwise interactions between organisms (vertices), and $u$ is a vertex of $G$ playing strategy $s_u$, then the fitness of $u$ is given by

$$F(u) = \frac{\sum_{v \in \Gamma(u)} F(s_u | s_v)}{|\Gamma(u)|}.$$

Here $s_v$ is the strategy being played by the neighbor $v$, and $\Gamma(u) = \{v \in V : (u, v) \in E\}$. One can view the fitness of $u$ as the average fitness $u$ would obtain if it played each of its neighbors, or the expected fitness $u$ would obtain if it were assigned to play one of its neighbors chosen uniformly at random.

Classical evolutionary game theory examines an infinite, symmetric population. Graphs or networks are inherently finite objects, and we are specifically interested in their asymmetries, as discussed above. Thus all of our definitions shall revolve around an infinite family $G = \{G_n\}_{n=0}^{\infty}$ of finite graphs $G_n$ over $n$ vertices, but we shall examine asymptotic (large $n$) properties of such families.

We first give a definition for a family of mutant vertex sets in such an infinite graph family to *contract*.

**Definition 29.13**    Let $G = \{G_n\}_{n=0}^{\infty}$ be an infinite family of graphs, where $G_n$ has $n$ vertices. Let $M = \{M_n\}_{n=0}^{\infty}$ be any family of subsets of vertices of the $G_n$ such that $|M_n| \geq \epsilon n$ for some constant $\epsilon > 0$. Suppose all the vertices of $M_n$ play a common (mutant) strategy $t$, and suppose the remaining vertices in $G_n$ play a common (incumbent) strategy $s$. We say that $M_n$ *contracts* if for sufficiently large $n$, for all but $o(n)$ of the $j \in M_n$, $j$ has an incumbent neighbor $i$ such that $F(j) < F(i)$.

A reasonable alternative would be to ask that the condition above holds for *all* mutants rather than all but $o(n)$. Note also that we only require that a mutant have *one* incumbent neighbor of higher fitness in order to die; one might consider requiring more. In Section 29.6 we ask the reader to consider one of these stronger conditions and demonstrate that our results can no longer hold.

To properly define an ESS for an infinite family of finite graphs in a way that recovers the classical definition asymptotically in the case of the family of complete graphs, we first must give a definition that restricts attention to families of mutant vertices that are smaller than some invasion threshold $\epsilon' n$, yet remain some constant fraction of the population. This prevents "invasions" that survive merely by constituting a vanishing fraction of the population.

**Definition 29.14**     Let $\epsilon' > 0$, and let $G = \{G_n\}_{n=0}^{\infty}$ be an infinite family of graphs, where $G_n$ has $n$ vertices. Let $M = \{M_n\}_{n=0}^{\infty}$ be any family of (mutant) vertices in $G_n$. We say that $M$ is $\epsilon'$-*linear* if there exists an $\epsilon$, $\epsilon' > \epsilon > 0$, such that for all sufficiently large $n$, $\epsilon' n > |M_n| > \epsilon n$.

We can now give our definition for a strategy to be evolutionarily stable when employed by organisms interacting with their neighborhood in a graph.

**Definition 29.15**     Let $G = \{G_n\}_{n=0}^{\infty}$ be an infinite family of graphs, where $G_n$ has $n$ vertices. Let $F$ be any 2-player, symmetric game for which $s$ is a strategy. We say that $s$ is an ESS with respect to $F$ and $G$ if for all mutant strategies $t \neq s$, there exists an $\epsilon_t > 0$ such that for any $\epsilon_t$-linear family of mutant vertices $M = \{M_n\}_{n=0}^{\infty}$ all playing $t$, for $n$ sufficiently large, $M_n$ contracts.

Thus, to *violate* the ESS property for $G$, one must witness a family of mutations $M$ in which each $M_n$ is an arbitrarily small but nonzero constant fraction of the population of $G_n$, but does not contract (i.e., every mutant set has a subset of linear size that survives all of its incumbent interactions). One can show that the definition given coincides with the classical one in the case where $G$ is the family of complete graphs, in the limit of large $n$. We note that even in the classical model, small sets of mutants were allowed to have greater fitness than the incumbents, as long as the size of the set was $o(n)$.

In the definition above there are three parameters: the game $F$, the graph family $G$, and the mutation family $M$. Our main results will hold for *any* 2-player, symmetric game $F$. We will study a rather general setting for $G$ and $M$: that in which $G$ is a family of random graphs and $M$ is arbitrary. We will see that, subject to conditions on degree or edge density (essentially forcing connectivity of $G$ but not much more), for *any* 2-player, symmetric game, the ESS of the classical settings, and only those strategies, are always preserved. Thus, for the purposes of characterizing stable strategies, the classical method of pairing organisms at random, is equivalent to randomizing the graph.

### 29.4.1  Random Graphs, Adversarial Mutations

We now proceed to state and prove the random graph result in the network ESS model. We consider a setting in which the graphs are generated via the $G_{n,p}$ model of Erdös and

Rényi. In this model, every pair of vertices is joined by an edge independently and with probability $p$ (where $p$ may depend on $n$). The mutant set, however, will be constructed adversarially (subject to the linear size constraint given by Definition 29.15). For this setting, we show that for any 2-player, symmetric game, $s$ is a classical ESS of that game, if and only if $s$ is an ESS for $\{G_{n,p}\}_{n=0}^{\infty}$, where $p = \Omega(1/n^c)$ and $0 \leq c < 1$, and any mutant family $\{M_n\}_{n=0}^{\infty}$, where each $M_n$ has linear size. We note that under these settings, if we let $c = 1 - \gamma$ for small $\gamma > 0$, the expected number of edges in $G_n$ is $n^{1+\gamma}$ or larger – that is, just superlinear in the number of vertices and potentially far smaller than $O(n^2)$. It is easy to convince oneself that once the graphs have only a linear number of edges, we are flirting with disconnectedness, and there may simply be large mutant sets that can survive in isolation due to the lack of any incumbent interactions in certain games. Thus in some sense we examine the minimum plausible edge density.

**Theorem 29.16** *Let $F$ be any 2-player, symmetric game, and suppose $s$ is a classical ESS of $F$. Let the infinite graph family $G = \{G_n\}_{n=0}^{\infty}$ be drawn according to $G_{n,p}$, where $p = \Omega(1/n^c)$ and $0 \leq c < 1$. Then with probability 1, $s$ is an ESS with respect to $F$ and $G$.*

A central idea in the proof is to divide mutants into two categories, those with "normal" fitness and those with "abnormal" fitness. Normal fitness means within a $(1 \pm \tau)$ factor of the fitness given by the classical model, where $\tau$ is a small constant greater than 0, and abnormal fitness means outside of that range. We will use the lemma below (provided without proof) to bound the number of incumbents and mutants of abnormal fitness.

**Lemma 29.17** *For almost every graph $G_{n,p}$ with $(1 - \epsilon)n$ incumbents, all but $\frac{24 \log n}{\tau^2 p}$ incumbents have fitness in the range $(1 \pm \tau)[(1 - \epsilon)F(s|s) + \epsilon F(s|t)]$, where $p = \Omega(1/n^c)$ and $\epsilon$, $\tau$ and $c$ are constants satisfying $0 < \epsilon < 1$, $0 < \tau < 1/6$, $0 \leq c < 1$. Similarly, under the same assumptions, all but $\frac{24 \log n}{\tau^2 p}$ mutants have fitness in the range $(1 \pm \tau)[(1 - \epsilon)F(t|s) + \epsilon F(t|t)]$.*

With this lemma we first show that all but $o(n)$ of the population (incumbent or mutant) have an incumbent neighbor of normal fitness. This will imply that all but $o(n)$ of the mutants of normal fitness have an incumbent neighbor of *higher* fitness. The vehicle for proving this is the following result from random graph theory, which gives an upper bound on the number of vertices not connected to a sufficiently large set, $U$.

**Theorem 29.18** *Suppose $\delta = \delta(n)$ and $C = C(n)$ satisfy $\delta pn \geq 3 \log n$, $C \geq 3 \log(e/\delta)$, and $C\delta n \to \infty$. Then almost every $G_{n,p}$ is such that for every $U \subset V$, $|U| = u = \lceil C/p \rceil$ the set $T_u = \{x \in V \setminus U \mid \Gamma(x) \cap U = \emptyset\}$ has at most $\delta n$ elements.*

This theorem assumes that the size of this large set $U$ is known with equality, which necessitates the union bound argument below. The second main step of the proof uses Lemma 29.17 again, to show that there can be at most $o(n)$ mutants with abnormal fitness. Since there are so few of them, even if none of them have an incumbent neighbor of higher fitness, $s$ will still be an ESS with respect to $F$ and $G$.

**PROOF** (Sketch)   Let $t \neq s$ be the mutant strategy. Since $s$ is a classical ESS, there exists an $\epsilon_t$ such that $(1 - \epsilon)F(s|s) + \epsilon F(s|t) > (1 - \epsilon)F(t|s) + \epsilon F(t|t)$, for all $0 < \epsilon < \epsilon_t$. Let $M$ be any mutant family that is $\epsilon_t$-linear. Thus for any fixed value of $n$ that is sufficiently large, there exists an $\epsilon$ such that $|M_n| = \epsilon n$ and $\epsilon_t > \epsilon > 0$. Also, let $I_n = V_n \setminus M_n$ and let $I' \subseteq I_n$ be the set of incumbents that have fitness in the range $(1 \pm \tau)[(1 - \epsilon)F(s|s) + \epsilon F(s|t)]$ for some constant $\tau$, $0 < \tau < 1/6$. Lemma 29.17 shows $(1 - \epsilon)n \geq |I'| \geq (1 - \epsilon)n - \frac{24 \log n}{\tau^2 p}$. Finally, let

$$T_{I'} = \{x \in V \setminus I' \mid \Gamma(x) \cap I' \neq \emptyset\}.$$

(For the sake of clarity we suppress the subscript $n$ on the sets $I'$ and $T$.) The union bound gives us

$$\Pr(|T_{I'}| \geq \delta n) \leq \sum_{i=(1-\epsilon)n-\frac{24 \log n}{\tau^2 p}}^{(1-\epsilon)n} \Pr(|T_{I'}| \geq \delta n \text{ and } |I'| = i). \qquad (29.7)$$

Letting $\delta = n^{-\gamma}$ for some $\gamma > 0$ gives $\delta n = o(n)$. We will apply Theorem 29.18 to the summand on the-right hand side of Equation 29.7. If we let $\gamma = (1 - c)/2$, and combine this with the fact that $0 \leq c < 1$, all of the requirements of this theorem will be satisfied (details omitted). Now when we apply this theorem to Equation 29.7, we get

$$\Pr(|T_{I'}| \geq \delta n) \leq \sum_{i=(1-\epsilon)n-\frac{24 \log n}{\tau^2 p}}^{(1-\epsilon)n} \exp\left(-\frac{1}{6}C\delta n\right) \qquad (29.8)$$
$$= o(1).$$

This is because Equation 29.8 has only $\frac{24 \log n}{\tau^2 p}$ terms, and Theorem 29.18 gives us that $C \geq (1 - \epsilon)n^{1-c} - \frac{24 \log n}{\tau^2}$. Thus we have shown, with probability tending to 1 as $n \to \infty$, at most $o(n)$ individuals are not attached to an incumbent which has fitness in the range $(1 \pm \tau)[(1 - \epsilon)F(s|s) + \epsilon F(s|t)]$. This implies that the number of mutants of approximately normal fitness, not attached to an incumbent of approximately normal fitness, is also $o(n)$.

Now those mutants of approximately normal fitness that *are* attached to an incumbent of approximately normal fitness have fitness in the range $(1 \pm \tau)[(1 - \epsilon)F(t|s) + \epsilon F(t|t)]$. The incumbents that they are attached to have fitness in the range $(1 \pm \tau)[(1 - \epsilon)F(s|s) + \epsilon F(s|t)]$. Since $s$ is an ESS of $F$, we know $(1 - \epsilon)F(s|s) + \epsilon F(s|t) > (1 - \epsilon)F(t|s) + \epsilon F(t|t)$, thus if we choose $\tau$ small enough, we can ensure that all but $o(n)$ mutants of normal fitness have a neighboring incumbent of higher fitness.

Finally by Lemma 29.17, we know that there are at most $o(n)$ mutants of abnormal fitness. So even if all of them are more fit than their respective incumbent neighbors, we have shown all but $o(n)$ of the mutants have an incumbent neighbor of higher fitness.   $\square$

Next we briefly outline how to prove a converse to Theorem 29.16. Observe that if in the statement of Theorem 29.16 we let $c = 0$, then $p = 1$, which in turn, makes $G =$

$\{K_n\}_{n=0}^{\infty}$, where $K_n$ is a clique of $n$ vertices. Then for any $K_n$ all of the incumbents will have identical fitness and all of the mutants will have identical fitness. Furthermore, if $s$ is an ESS for $G$, the incumbent fitness will be higher than the mutant fitness. Finally, one can show that as $n \to \infty$, the incumbent fitness converges to $(1 - \epsilon)F(s|s) + \epsilon F(s|t)$, and the mutant fitness converges to $(1 - \epsilon)F(t|s) + \epsilon F(t|t)$. In other words, $s$ must be a classical ESS, providing a converse to Theorem 29.16.

## 29.5 Future Work

Most evolutionary game-theoretic models consider an infinite population of agents. These agents usually obey some simple dynamic such as imitation or replication. Typical results in these models show that in the limit (as time goes to infinity) the population converges to an equilibrium. A major open problem in the intersection of evolutionary game theory and theoretical computer science is to analyze a population of $n$ agents, who obey one of these dynamics, and bound the time of convergence to an equilibrium. The notions of equilibrium and stability might have to be adapted to this new finite setting. Results along these lines would yield simple, distributed algorithms that agents could implement and converge to an equilibrium in a bounded (and hopefully short) amount of time. This would provide contribution beyond proving the existence of equilibria, and beyond showing that an infinite population will eventually converge to it. It will show that a population of a given size will converge to a stable equilibrium within a certain amount of time.

To start on this endeavor, the simplest models could consider $n$ agents, where each agent could interact with each other agent. One example of such a problem would be to analyze a selfish routing model, such as the one described in Section 29.3, except with $n$ agents, as opposed to infinitely many, and show a strongly polynomial time bound for their convergence. After baseline models such as this have been developed and studied, one might then try to find dynamics that result in these agents converging to an equilibrium that maximizes an appropriate notion of social welfare. Another extension would be to consider models where agents are arranged in a graph and can only interact with agents in their local neighborhood. One could then analyze not only the effect of the graph topology on equilibrium, as was done in Section 29.4, but also how it affects the convergence time.

It may turn out that hardness results stand in the way of such progress. Then one could try to bound the time of convergence to an approximate equilibrium, or simply bound the amount of time the population spends far away from an equilibrium. Also results such as the one given in Section 29.2 imply that there exist games for which it is hard to compute equilibria. There still could be many well-motivated classes of games for which arriving at an equilibrium is computationally tractable.

## 29.6 Notes

The motivation for evolutionary game theory and the description of the model, definitions, and dynamics were inspired by Smith (1982), Osborne and Rubinstein (1994), Weibull (1995), Hofbauer and Sigmund (1998), Kontogiannis and Spirakis (2005),

and Kearns and Suri (2006). The Hawks and Doves game and its motivation come from Smith (1982), Osborne and Rubinstein (1994), Weibull (1995), and Alexander (2003).

The section on the computational complexity of ESS comes from Nisan (2006), which extended work by Etessami and Lochbihler (2004). Lemma 29.3 is a slight modification of a lemma in Motzkin and Straus (1965). Papadimitriou and Yannakakis (1982) show the problem of determining whether or not a graph has a maximum clique of size $k$ is $coD^p$-hard. We will not define the complexity class $coD^p$ here, but simply state that it contains both NP and coNP. Etessami and Lochbihler (2004) show that finding a strategy that is close in $\ell_p$ norm to and ESS takes super-polynomial time unless P=NP. They also show that finding an ESS is in $\Sigma_2^P$, and that finding a regular ESS is NP-complete. In addition, they prove that counting the number of ESS and counting the number of regular ESS are both #P-hard.

Most of Section 29.3 comes from Fischer and Vöcking (2004) and Fischer (2005). For more details regarding the convergence of the imitative dynamics to a Nash flow, see those two references. We refer the reader to Brauer and Nohel (1969) for an excellent introduction into the Lyapunov framework. For a more extensive and technical treatment see Bhatia and Szegö (1970). For applications of the Lyapunov framework to other evolutionary game theoretic models and dynamics, see Weibull (1995) and Hofbauer and Sigmund (1998). There are many other places where evolutionary game theory is studied in conjunction with imitative dynamics, for example see Björnerstedt and Schlag (1996) and Schlag (1998) and chapter 4 of Weibull (1995).

There is a nice sequence of papers that continues the work of Fischer and Vöcking (2004) shown in Section 29.3. Fischer and Vöcking (2005) consider a similar model where agents may have stale information regarding the latencies of other paths. Fischer et al. (2006) consider a model where agents switch paths in a round based fashion.

Section 29.4 comes from Kearns and Suri (2006) . Vickery (1987) first noticed that a constant number of mutants may have higher fitness than the incumbents who are playing an ESS. Theorem 29.18 is Theorem 2.15 from Bollobás (2001) . In Kearns and Suri (2006), the authors give a pair of results dual to Theorem 29.16 and its converse. They show that if the graph is chosen adversarially, subject to some density restrictions, and the mutants are chosen randomly then ESS are preserved.

## Acknowledgments

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## Exercises

**29.1** Find the ESS of Prisoners Dilemma.

**29.2** In the game of Hawks and Doves, given by Figure 29.1, if $V \leq C$, show that $V/C$ is a mixed strategy ESS. (Hint: Use the fact that for any mixed Nash equilibrium, $s^*$ with support $s_1, s_2, \ldots, s_k$, $F(s_1|s^*) = F(s_2|s^*) = \cdots = F(s_k|s^*) = F(s^*|s^*)$.)

**29.3** Consider a $2 \times 2$-symmetric game with four arbitrary constants for payoffs. Characterize the ESS for such a game in terms of the payoffs. Use this to conclude that any $2 \times 2$-symmetric game has an ESS.

**29.4** Give an example of a game that has a Nash Equilibrium but no ESS.

**29.5** Prove Lemma 29.7.

**29.6**  Show that $\sum_{p \in P} x'_p = 0$, where $x'_p$ is defined by Equation 29.2. Using this, conclude that if, in the selfish routing model of Section 29.3, the imitative dynamics initially start with a feasible flow, then for all time the flow remains feasible.

**29.7**  Show that there exists a game such that with high probability for a family of random graphs with $p = \Omega(1/n^c)$ and $0 \leq c < 1$, an adversary can construct a mutant set such that there will exist at least one mutant with higher fitness than all of its incumbent neighbors.